FINAL EXAM ISI BANGALORE APRIL 2016

B MATH ALGEBRA II 100 POINTS

Notes.

(a) Justify all your steps. You may use any result proved in class unless you have been asked to prove the same.

(b) $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers, $\mathbb{R}$ = real numbers, $\mathbb{C}$ = complex numbers, $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$.

(c) By default, F denotes a field.

---

1. [8 points] Give an example of a 2×2 matrix over $\mathbb{Q}$ which is not upper triangulable over $\mathbb{R}$.

2. [27 points] For the matrix A given below, find the characteristic polynomial, eigenvalues and eigenvectors (over $\mathbb{C}$ if necessary). Find an invertible 3×3 matrix X such that XAX^{-1} is a diagonal matrix. Using this, compute A^{100}.

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -10 & 11 & -1 \\ -10 & 10 & 0 \end{pmatrix}$$

3. [13 points] Give an example of 2 square matrices A, B of the same size such that the characteristic polynomials of A, B are equal and their minimal polynomials are also equal, but A is not similar to B. Justify your answer.

4. [13 points] Prove that over any field F, and for any integer $n > 0$, an $n \times n$ matrix of rank 1 is upper-triangulable. Show that for any n, there is an $n \times n$ matrix of rank 1 which is not diagonalizable.

5. [13 points] Find the 3×3 matrix A over $\mathbb{R}$ such that multiplication by A induces a rotation of $\mathbb{R}^3$ with axis $[1, 0, 1]^t$ and angle $\pi/2$ (with respect to a positively oriented orthonormal basis).

6. [13 points] In each of the following cases give an example of a symmetric 2×2 matrix A over $\mathbb{R}$ having all nonzero entries satisfying the property given below. Also find an orthogonal basis for the symmetric bilinear form on $\mathbb{R}^2$ given by $\langle v, w \rangle := v^t A w$.
   (i) A is positive definite.
   (ii) A is neither positive definite nor negative definite.

7. [13 points] Supply proofs of the following statements proved in class.
   (i) Any real symmetric matrix has real eigenvalues.
   (ii) If A is a normal matrix over $\mathbb{C}$ and $v \in \mathbb{C}^n$ is an eigenvector, i.e., $Av = \lambda v$, then $A^* v = \bar{\lambda} v$.